Exercises (Let f be $\uparrow$ on $[a,b]$ & $\begin{cases} f(x) = f(b) \ \forall x > b \\ f(x) = f(a) \ \forall x < a \end{cases}$).

1. $A = f^{-1}((\alpha, +\infty))$ is an interval because $x_1, x_2 \in A$ & $x_1 \leq x \leq x_2$ imply $x \in A$.

2. $x \mapsto f(x)$ & $x \mapsto f(x+c)$ are measurable

3. Let $x \in (a,b) \cap G$ with some open G and let $D_- f(x) := \underline{\lim}_{\substack{x' \to x \\ x' < x}} \frac{f(x) - f(x')}{x - x'} = \sup_{\delta > 0} \left(\inf \left\{ \frac{f(x) - f(x')}{x - x'} : 0 < x - x' < \delta \right\} \right)$,

and suppose that $D_- f(x) < \alpha$ $(\in \mathbb{R})$. Let $\mathcal{I}_x$ consist of all nondegenerate intervals contained in $(a,b) \cap G$ and of the form $[x', x]$ such that $\frac{f(x) - f(x')}{x - x'} < \alpha$. Show that $\mathcal{I}_x$ is a Vitali cover of the singleton $\{x\}$.

4. Let $J_j := [y_j, y_j']$ $(j = 1, 2, \dots n)$ be non-degenerate disjoint intervals contained in $[c,d]$ $(\subseteq [a,b])$. Show that
$$\sum_{j=1}^{n} \left(f(y_j') - f(y_j) \right) \leq f(d) - f(c) \qquad (\text{Hint: } f \uparrow)$$

5. Let $g \in \mathcal{L}[a+c, b+c]$. Show that
$$(*) \quad \int_{a+c}^{b+c} g(z)\, dz = \int_a^b g(x+c)\, dx$$
progressively for
i) $g = \chi_E$ with $E \in \mathfrak{M}$
ii) $g \in \mathcal{S}_0$
iii) $0 \leq g \in \mathcal{L}[a+c, b+c]$
(iv) general $g \in \mathcal{L}[a+c, b+c]$.

limit for i). In (*),

$$\text{LHS} = \int_{[a+c,\, b+c]} \chi_E = m\Big([a+c, b+c] \cap E\Big) = m\Big([a,b] \cap (E-c)\Big)$$

$$= \int_a^b \chi_{E-c}(z)\, dz = \int_a^b \chi_E(z+c)\, dz = \text{RHS}$$

6. Show that $x \mapsto D_- f(x)$ is a non-negative extended-real valued function on (a,b); also that for $D^+ f(x)$ etc.

7. Let $0 < \alpha < \beta < +\infty$ and

$$E_{\alpha,\beta} = \{ x \in (a,b) : D_- f(x) < \alpha < \beta < D^+ f(x) \}$$

Show that $m^*(E_{\alpha,\beta}) = 0$ via the following route: $\forall \varepsilon$, take open G with $E_{\alpha,\beta} \subseteq G \subseteq (a,b)$ s.t. $m(G) < m^*(E_{\alpha,\beta}) + \varepsilon$. By Q3 & Vitali Th, take disjoint (non-degenerate) intervals contained in G and of the form $I_i := [x_i', x_i]$ $(i = 1, 2, \cdots N)$ s.t.

$$\Big(f(x_i) - f(x_i')\Big) < \alpha \cdot l(I_i) = \alpha(x_i - x_i') \quad (i = 1, 2, \cdots N)$$

and

$$m^*(E_{\alpha,\beta}) - \varepsilon < m^*\Big(E_{\alpha,\beta} \cap \bigcup_{i=1}^{N} I_i\Big).$$

Let I_i° denote the interior of I_i and $A := E_{\alpha,\beta} \cap \bigcup_{i=1}^{N} I_i^\circ$. Then

1. $m^*(E_{\alpha,\beta}) - \varepsilon < m^*(A)$
2. $\sum_{i=1}^{N} (f(x_i) - f(x_i')) < \sum_{i=1}^{N} \alpha \cdot l(I_i) \leq \alpha\, m(G) < \alpha\big(m^*(E_{\alpha,\beta}) + \varepsilon\big)$

Let $y \in A$ and let $\mathcal{J}_y$ denote the family of all intervals $[y, y']$ contained in some I_i° such that

$$\beta < \frac{f(y') - f(y)}{y' - y}.$$

Since $\beta < D^+ f(y)$, $\mathcal{J}_y$ is a Vitali cover of $\{y\}$, so $\mathcal{J} := \bigcup\{\mathcal{J}_y : y \in A\}$ is a Vitali Cover of A & so

$\exists$ finitely many disjoint $J_j := [y_j, y_j'] \in \mathcal{G}$ $(j = 1, 2, \cdots, M)$ such that $m^*(A) - \varepsilon < m^*(A \cap \bigcup_{j=1}^{M} J_j) \left(\leq \sum_{j=1}^{M} \ell(J_j)\right)$. Noting each J_j is contained in some I_i^o and since $f \uparrow$ one has from Q4 that

$$③ \quad \sum_{j=1}^{M} (f(y_j') - f(y_j)) \leq \sum_{i=1}^{N} \sum_{[y_j, y_j'] \subseteq [x_i', x_i]} (f(y_j') - f(y_j)) \leq \sum_{i=1}^{N} (f(x_i) - f(x_i')) \quad ;$$

also, since $\beta \ell(J_j) = \beta(y_j' - y_j) < f(y_j') - f(y_j)$, it follows that

$$\beta\left(m^*(A) - \varepsilon\right) \leq \beta \sum_{j=1}^{M} \ell(J_j) < \sum_{j=1}^{M} (f(y_j') - f(y_j)) \leq \sum_{i=1}^{N} \left(f(x_i) - f(x_i')\right)$$

and from ① + ② that

$$\beta\left(m^*(E_{\alpha,\beta}) - 2\varepsilon\right) \leq \alpha\left(m^*(E_{\alpha,\beta}) + \varepsilon\right)$$

Since $\varepsilon > 0$ is arbitrary, $\beta(m^*(E_{\alpha,\beta})) \leq \alpha(m^*(E_{\alpha,\beta}))$ and so $m^*(E_{\alpha,\beta}) = 0$ because $\alpha < \beta$.

8. By Q7, $f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{n} \frac{f(x + 1/n) - f(x)}{1/n}$ exists a.e. $x \in (a, b)$. Show that $f'(x) \geq 0$ a.e. and f' measurable. By Q5 and Fatou (applied to (f_n) where $f_n : x \mapsto \frac{f(x + 1/n) - f(x)}{1/n}$ $\forall x \in [a, b]$ (recall that $f(x) := f(b)$ $\forall x > b$), show that $\int_a^b f' \leq f\big|_a^b$.

(Hint: $n \int_b^{b+1/n} f = f(b)$ & $\int_a^{a+1/n} f \geq f(a)$ as $f \uparrow$).